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Irrelevant Operators and Momentum-Shell Recursion Relations in $d = 2 + \epsilon$ Dimensions

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The momentum-shell recursion relations of Nelson and Pelcovits for an n-vector model near two dimensions are reexamined. The renormalization of the infinite set of relevant and marginal operators present in the system is studied. Ambiguities obtained in the ensuing recursion relations are shown to involve irrelevant operators only, thus justifying the procedure of Nelson and Pelcovits. The cases of finite external field h and finite spin anisotropy g are both considered.

KEY WORDS: Renormalization group; magnetism; anisotropy.

1. INTRODUCTION

The critical behavior of n-component classical spin systems was studied by various authors[1-5] near two dimensions, using a spin-wave expansion originally due to Berezinskii and Blank.[6] In this method the $O(n)$ symmetry of the starting Hamiltonian is broken formally at the outset, and the properties of the system under renormalization are studied. Nelson and Pelcovits[4] have employed the momentum-shell technique of Wilson and Kogut[7] to construct a simple and physically appealing version of the theory, in which the Hamiltonian flows are directly mapped out, in analogy with the procedure employed near four dimensions.[7,8] The present case is more complicated than the four-dimensional one, however, due to the existence of arbitrary numbers of relevant and marginal operators for $d = 2$. Indeed, the spin field $s(x)$ has dimension $[s] = d/2 - 1$, so that terms in the Hamiltonian containing high powers of s, which are irrelevant near four dimensions, are relevant[4] for $d = 2$.

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When $d \lesssim 4$, the coefficient of the relevant term s^2 is adjusted by fixing the temperature, and the coefficients of the marginal operators $(\partial_\mu s)^2$ and s^4 are controlled by the renormalization procedure. Near two dimensions, on the other hand, the relevant terms s^{2m} are fixed by the constraint

$$s^2(x) = 1 \tag{1.1}$$

and one is left with the marginal term $(\partial_\mu s)^2$. The fixed-spin condition (1.1) now imposes a nonlinear restriction on fluctuations of the field $s(x)$, which makes a direct application of perturbation theory difficult. To circumvent this problem a spin-wave expansion is carried out,[6] by aligning the system along one of the spin components, $\sigma(x)$, and expanding in the transverse fluctuations of the $(n-1)$ component vector $\pi(x)$. The marginal term $(\partial_\mu s)^2$ then gives rise to an infinite series generated by the constraint (1.1):

$$(\partial_\mu s)^2 = (\partial_\mu \pi)^2 + (\partial_\mu \sigma)^2 = (\partial_\mu \pi)^2 + (1-\pi^2)^{-1}(\pi \cdot \partial_\mu \pi)^2 \tag{1.2}$$

$$= (\partial_\mu \pi)^2 + (\pi \cdot \partial_\mu \pi)^2 + \pi^2(\pi \cdot \partial_\mu \pi)^2 + \cdots \tag{1.3}$$

The $O(n)$ symmetry of the original system is reflected in the fact that all the terms in (1.3) have the same coefficient.

In studying the renormalization group near two dimensions[1,3,4] one therefore has an infinite number of (marginal) terms in the effective Hamiltonian which are equally relevant. Nelson and Pelcovits[4] found recursion relations for the Hamiltonian

$$H = \int d^dx \left\{ \left(-\frac{1}{2T}\right)\left[(\partial_\mu \pi)^2 + (\partial_\mu \sigma)^2\right] + \frac{h}{T}\sigma \right\} \tag{1.4a}$$

$$= \int d^dx \left\{ \left(-\frac{1}{2T}\right)\left[(\partial_\mu \pi)^2 + (\pi \cdot \partial_\mu \pi)^2 + \pi^2(\pi \cdot \partial_\mu \pi)^2 + \cdots \right] \right.$$
$$\left. + \frac{h}{T}\left[1 - \frac{1}{2}\pi^2 - \frac{1}{8}\pi^4 + \cdots \right] \right\} \tag{1.4b}$$

by studying the renormalization of terms in (1.4b) containing two powers of π, i.e., $(\partial_\mu \pi)^2$ and π^2. It is natural to ask, however, whether the same results would be obtained if one examined the renormalization of the equally relevant quantities $(\pi \cdot \partial_\mu \pi)^2$ and π^4, for instance, or any of the other terms in (1.4b). It turns out that for finite h different recursion relations are in fact obtained, depending on which quantities are examined, since the form of the Hamiltonian (1.4) and the fixed-spin constraint (1.1) are not preserved under renormalization.

In order to understand this particular multiplicity (which does not occur near $d = 4$) we have examined the irrelevant operators of the theory, which are generated by an expansion of the renormalized Hamiltonian (1.4) in the magnetic field h. To lowest order in h recursion relations are found which do not involve any irrelevant operators. These yield the correct equation of state to order ϵ. In order to calculate corrections to scaling one needs the leading (dimension four) irrelevant operators, which involve terms in h and h^2.

The nonlinear recursion relations of Nelson and Pelcovits[(4)] correspond to the inclusion of an infinite set of irrelevant terms in the "lowest-order" Hamiltonian. The equation of state obtained from these recursion relations, as well as from the other possible renormalization schemes mentioned above, are also correct to lowest order in ϵ. Corrections to scaling, on the other hand, will not be calculable in this way, due to the unsystematic treatment of irrelevant operators.

In the presence of spin anisotropy the system crosses over from $O(n)$ to $O(1)$ (Ising), or $O(n-1)$ symmetry depending on the sign of the anisotropy. In the former case the new fixed point lies outside the range of validity of the ϵ expansion, but in the latter case it remains at low temperature [$T^* = O(\epsilon)$ for $n > 3$]. It is then important to find nonlinear recursion relations which correctly interpolate between the limits of small and large effective anisotropy. Once again, there are many equivalent sets of recursion relations which accomplish this task, and those of Ref. 4 offer one such example. [A linearization about the $O(n)$ system, analogous to the one we carry out in the presence of a magnetic field, will of course not yield the full crossover.]

Our study illuminates the considerable freedom which exists in the choice of renormalization group variables, especially near $d = 2$. The Hamiltonian flows have topological significance, reflecting the relative stability of different fixed points, but their metric properties are largely dependent on arbitrary definitions.

Section 2 discusses the $O(n)$ model with and without a magnetic field, and Section 3 studies the case of spin anisotropy. In the Appendix we briefly discuss the evaluation of diagrams in the presence of a magnetic field or spin anisotropy, for terms involving derivative coupling. A fuller discussion will be presented elsewhere.[(9)]

2. THE n-VECTOR MODEL

2.1. Zero External Magnetic Field ($h = 0$)

Our motivation and formalism are the same as in Ref. 4, and we refer the reader to that paper for a more detailed discussion. [Unless otherwise noted,[3] we shall adopt the same notation as in Ref. 4.] The $O(n)$

[3] Our changes in notation, with respect to Ref. 4, are as follows: we write H instead of $\overline{H}_1$, and drop the arrows on the vector fields such as π. For finite anisotropy, we put g in the direction π_1 and keep $\sigma^2 = 1 - \pi^2$, with $\pi = (\pi_1, \tilde{\pi})$. Finally, we keep Λ explicit, i.e., we do not set $\Lambda = 1$ in evaluating diagrams.

model

$$H[\pi,\sigma] = \int d^dx\left(-\frac{1}{2T}\right)\left[(\partial_\mu\pi)^2 + (\partial_\mu\sigma)^2\right] \tag{2.1}$$

with fixed-spin constraint

$$\pi^2(x) + \sigma^2(x) = 1 \tag{2.2}$$

can be transformed into a model for the $(n-1)$-component π field, by eliminating the scalar σ.

$$H[\pi] = \int d^dx\left\{\left(-\frac{1}{2T}\right)\left[(\partial_\mu\pi)^2 + \frac{(\pi\cdot\partial_\mu\pi)^2}{1-\pi^2}\right] - \frac{\rho}{2}\ln(1-\pi^2)\right\} \tag{2.3}$$

where

$$\rho = (2\pi)^{-d}\int_0^\Lambda d^dq$$

is the number of degrees of freedom per unit volume in the spherical Brillouin zone, and Λ is the upper wave number cutoff [in Eqs. (2.1)–(2.3) we only consider fields varying on wavelengths longer than Λ^{-1}]. The last term in Eq. (2.3) reflects the fixed spin constraint (2.2). When (2.3) is expanded in π^2 one obtains an infinite set of terms of the form $\rho\pi^{2(m+1)}/(m+1)$ and $T^{-1}\pi^{2m}(\pi\cdot\partial_\mu\pi)^2$, $(m = 0, 1, 2 \ldots)$ with the same coefficients ρ and T^{-1}, reflecting the $O(n)$ symmetry of the starting model (2.1).

The renormalization group transformation is now defined in the usual way[7,4] by integration of the degrees of freedom of $\pi(q)$ in the shell $\Lambda/b < q < \Lambda$ in momentum space, and rescaling of the field and wave vectors. Since the transformation involves the π field only, it is not obvious a priori that the Hamiltonian will retain the rotationally invariant form (2.1), especially for a system with finite cutoff Λ.[4] If the Hamiltonian does not retain the form (2.1), there is ambiguity in the definition of the renormalized temperature (or coupling constant) T'.

Using the momentum-shell technique it is essentially impossible to check the symmetry of the renormalized Hamiltonian H' to all orders in π^2, except in the limit $n\to\infty$. In that case, the graphs in Fig. 1a can all be

[4] As shown in Ref. 3, for a theory with infinite cutoff, the existence of Ward identities ensures that the renormalized theory preserves the $O(n)$ symmetry of the bare Hamiltonian. However, in a theory with finite cutoff we do not know how to prove the Ward identities because the transformation law under rotation for σ_q and π_q is highly nonlocal.

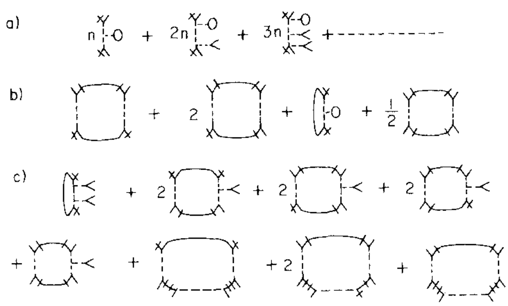

Fig. 1. (a) The series of graphs which contribute to the renormalization of $\sum_{m=0}^{\infty}\pi^{2m}$ $(\pi \cdot \partial_\mu \pi)^2$ in the case $n \to \infty$. The notation is the same as in Ref. 4. Solid lines are propagators, and slashes on lines indicate derivatives. Dashed lines separate pairs of spin with common indices. (b) and (c) represent graphs which contribute to the renormalization of 4-point and 6-point vertices, respectively. The net contributions from these graphs to $\pi^2(\partial_\mu \pi)^2$ and $\pi^4(\partial_\mu \pi)^2$ sum up to zero.

evaluated, and H' written in the form (2.3) with corrections containing four or more derivatives.

$$H' = \int d^d x \left\{ \left(-\frac{1}{2T'} \right) \left[(\partial_\mu \pi')^2 + \frac{(\pi' \cdot \partial_\mu \pi')^2}{1 - \pi'^2} \right] - \tfrac{1}{2} \rho \ln(1 - \pi'^2) + O(\partial_\mu^4) \right\} \tag{2.4}$$

where

$$\pi'_{q'} = \zeta^{-1} \pi_q \tag{2.5}$$

$$q' = bq \tag{2.6}$$

$$\zeta = b^d \left(1 - \frac{nT \ln b}{4\pi} \right) \tag{2.7}$$

$$T' = b^{-\epsilon} T \left(1 + \frac{nT \ln b}{2\pi} \right) \tag{2.8}$$

$$\epsilon = d - 2 \tag{2.9}$$

For finite n, the diagrams involving high powers of π are increasingly difficult to evaluate, so the form (2.3) must be inferred from the first few orders. For instance, if we calculate the renormalized temperature T' by comparing the coefficients of $(\partial_\mu \pi)^2$ and $(\pi \cdot \partial_\mu \pi)^2$ as in Ref. 4,[5] we find

$$\zeta = b^d \left(1 - \frac{n-1}{4\pi} T \ln b \right) \tag{2.10}$$

$$T' = b^{-\epsilon} T \left(1 - \frac{n-2}{2\pi} T \ln b \right) \tag{2.11}$$

[5] The cancellation shown in the top line of Fig. 2c of Ref. 4 is incorrect. The correct cancellation requires the slash on the lower left-hand external leg of the second graph to be on the lower right-hand external leg.

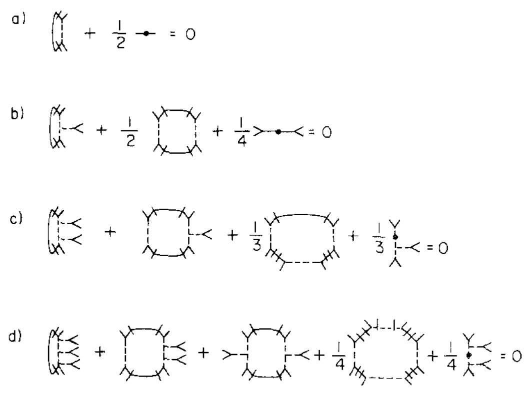

Fig. 2. The cancellations shown in parts (a)–(d) refer to the derivative-free terms in Eq. (2.3), of order π^2, π^4, π^6, and π^8, respectively. These cancellations lead to the preservation of the fixed-spin condition. The dots on the solid lines refer to vertices involving ρ.

which generalizes (2.7) and (2.8). A nontrivial check that H' really has the form (2.3) is obtained by calculating the coefficients of the terms such as $\pi^2(\partial_\mu\pi)^2$ or $\pi^4(\partial_\mu\pi)^2$, which are absent from (2.3). These are represented by the diagrams in Figs. 1b and 1c respectively, and turn out to vanish.[9] In addition the derivative-free terms in (2.5), examined to order π^8, are consistent[9] with the function $-\frac{1}{2}\rho\ln(1-\pi^2)$ appearing in (2.3) (see Fig. 2). From the above information we are led to the inference that similar behavior will persist to all orders in π, and that the form (2.3) is preserved for finite n also. The terms left out contain at least four derivatives and are irrelevant for the same reasons as near $d = 4$.[7]

2.2. The Case $h \neq 0$

Let us add the term $h\sigma/T$ to (2.1), i.e., consider the Hamiltonian

$$H = \int d^dx\left\{\left(-\frac{1}{2T}\right)\left[(\partial_\mu\pi)^2 + \frac{(\pi\cdot\partial_\mu\pi)^2}{1-\pi^2}\right]\right.$$
$$\left. -\frac{1}{2}\rho\ln(1-\pi^2) + \frac{h}{T}(1-\pi^2)^{1/2}\right\} \qquad (2.12)$$

If we repeat the calculation of the previous section we now find that the renormalized Hamiltonian is not consistent with the starting form (2.12).

Indeed, the renormalization of $(\partial_\mu \pi)^2$ is different from that of $(\pi \cdot \partial_\mu \pi)^2$ by h-dependent terms, and there are new terms generated of the form $\pi^{2m}(\partial_\mu \pi)^2$, which were not present earlier. There is thus some ambiguity in the parametrization of the renormalized Hamiltonian.

We shall follow the field-theoretic discussion(2,3) and order the terms generated according to their scaling dimension. (The magnetic field h has dimension two for $d = 2$, and the spin field as well as the temperature T are dimensionless.) A detailed calculation(9) shows that no additional dimension two operators are generated by the renormalization, other than those present in (2.12). We then identify the renormalized temperature T' from the coefficient of the term $[(\partial_\mu \pi)^2 + (1 - \pi^2)^{-1}(\pi \cdot \partial_\mu \pi)^2]$, and the renormalized field from the coefficient of $\sigma = 1 - \frac{1}{2}\pi^2 - \frac{1}{8}\pi^4 + \cdots$. From this identification it follows(9) that Eq. (2.11) is unchanged, whereas h' satisfies

$$\frac{h'}{T'} = \frac{h}{T}\zeta \tag{2.13}$$

where ζ is given by (2.10) as before. All the other terms generated involve higher powers of h and higher gradients. The operators of dimension four contain either h^2, $h\partial_\mu^2$ or ∂_μ^4 with arbitrary powers of the field π. Our calculation(9) shows that these are consistent with the form[6]

$$[H^{(4)}]' = \int d^d x \left\{ \sum_{i=1}^{5} \lambda_i^{(4)} \Theta_i^{(4)} \right\} \tag{2.14}$$

where Θ_1, Θ_2, and Θ_3 are terms with four derivatives,(3) and

$$\Theta_4 = \sigma^{-1}(h + \nabla^2\sigma)\left[(\partial_\mu \pi)^2 + (\partial_\mu \sigma)^2\right] \tag{2.15}$$

$$\Theta_5 = \sigma^{-2}(h + \nabla^2\sigma)^2 \tag{2.16}$$

It was shown in Ref. 3, using nonlinear Ward identities that the five operators $\Theta_1, \ldots, \Theta_5$ are the complete set of irrelevant operators of dimension four. (The physical significance of these irrelevant operators was discussed recently by Amit *et al.*(10).) Our calculation verifies (2.14), in that we calculate its expansion in π and check the first few (nontrivial) orders.(9)

[6] In the systematic study of irrelevant operators, it is useful to carry out the renormalization in a finite-momentum shell $\Lambda \geqslant q \geqslant \Lambda/b$. The irrelevant operators are singled out by their cutoff dependence which is different from that of the relevant operators. This distinction is lost if one uses an infinitesimal shell of unit radius as was done in Ref. 4.

The full renormalized Hamiltonian can thus be written in the form

$$H' = \overline{H}' + \overline{H}'_{\rm ir} \tag{2.17a}$$

$$\overline{H}' = \int d^d x \left\{ \left(-\frac{1}{2T'} \right) \left[(\partial_\mu \pi)^2 + \frac{(\pi \cdot \partial_\mu \pi)^2}{1 - \pi^2} \right] - \frac{\rho}{2} \ln(1 - \pi^2) + \frac{h'}{T'} (1 - \pi^2)^{1/2} \right\} \tag{2.17b}$$

$$\overline{H}'_{\rm ir} = \int d^d x \left\{ \sum_{m=2}^{\infty} \sum_{i=1}^{\nu_m} \lambda_i^{(2m)} \Theta_i^{(2m)} \right\} \tag{2.17c}$$

where $\Theta_i^{(2m)}$ is an irrelevant operator of dimension $2m$ (ν_m is the number of such operators at order $2m$). The irrelevant operators control corrections to scaling, but not the leading scaling functions. To see this, we write down the differential recursion relations for dimension two operators [cf. Eqs. (2.10), (2.11), (2.13)],

$$\frac{dT}{dl} = -\epsilon T(l) + \frac{(n-2)}{2\pi} T^2(l) \tag{2.18}$$

$$\frac{dh}{dl} = 2h(l) + \frac{(n-3)}{4\pi} h(l) T(l) \tag{2.19}$$

These are the linearized forms of the corresponding equations (2.16) of Nelson and Pelcovits.[4] We may repeat their derivation of the equation of state, given in Section V of Ref. 4, with Eq. (5.2) unchanged and (5.3) missing its second term. The ensuing modifications of Eqs. (5.7)–(5.10) are such as to yield precisely the same result as in (5.11) and (5.17) for the physical equation of state,

$$M(t,h) = \left[1 + \frac{(n-2)T}{2\pi\epsilon} (h^{\epsilon/2} - 1) \right]^{(n-1)/2(n-2)} \tag{2.20}$$

2.3. The Recursion Relations of Nelson and Pelcovits

Let us discuss briefly the relationship of the above symmetrical renormalization to the nonlinear scheme of Ref. 4. The Hamiltonian in the presence of h was represented in the form

$$H = H_0 + H_1 \tag{2.21}$$

$$H_0 = \int d^dx\left(-\frac{1}{2T}\right)\left[(\partial_\mu\pi)^2 + h\pi^2\right] \tag{2.22a}$$

$$H_1 = \int d^dx\left(-\frac{1}{2T}\right)\left[(\pi\cdot\partial_\mu\pi)^2 + \tfrac{1}{4}h\pi^4\right] \tag{2.22b}$$

and the renormalization of the coefficients in (2.22a) due to the perturbation (2.22b) calculated to linear order in T and to all orders in h. The spin rescaling $\bar{\zeta}$ was *defined* by the relation

$$\frac{\bar{h}'}{\bar{T}'} = \bar{\zeta}\frac{h}{T} \tag{2.23}$$

and became[3]

$$\bar{\zeta} = b^d\left[1 - \frac{(n-1)}{4\pi}\,\frac{T\ln b}{1 + h/\Lambda^2}\right] \tag{2.24}$$

The differential recursion relations turn out to be[(4)]

$$\frac{d\bar{T}}{dl} = -\epsilon\bar{T}(l) + \frac{(n-2)}{2\pi}\,\frac{\bar{T}^2(l)}{1+\bar{h}(l)} \tag{2.25}$$

$$\frac{d\bar{h}}{dl} = 2\bar{h}(l) + \frac{(n-3)}{4\pi}\,\frac{\bar{h}(l)\bar{T}(l)}{1+\bar{h}(l)} \tag{2.26}$$

where $\bar{h}(l)$ is a dimensionless variable and is equal to h/Λ^2.

The question naturally arises as to the renormalization of the other dimension two (relevant) operators occurring in H, for instance, the terms in H_1, Eq. (2.22b). Since their coefficients can be shown to obey different recursion relations, which only agree with (2.25) and (2.26) to lowest order in $h(l)$, the consistency of the procedure is not obvious a priori. In order to verify this consistency, we write the full renormalized Hamiltonian in the form

$$H' = \bar{\bar{H}}' + \bar{\bar{H}}'_{\text{ir}} \tag{2.27}$$

$$\bar{\bar{H}}' = \int d^dx\left\{\left(-\frac{1}{2\bar{T}'}\right)\left[(\partial_\mu\pi)^2 + \frac{(\pi\cdot\partial_\mu\pi)^2}{1-\pi^2}\right] + \frac{\bar{h}'}{\bar{T}'}(1-\pi^2)^{1/2} - \frac{\rho}{2}\ln(1-\pi^2)\right\} \tag{2.28a}$$

$$\overline{\overline{H}}_{\mathrm{ir}}' = \int d^dx \left\{ \sum_{m=0}^{\infty} \left[A_m(h) + B_m(h)(\partial_\mu \pi)^2 + C_m(h)(\pi \cdot \partial_\mu \pi)^2 + \cdots \right] \pi^{2m} + O(\partial_\mu^4) \right\} \tag{2.28b}$$

where $\overline{T}'$ and $\bar{h}'$ are the solutions of (2.25) and (2.26). The correction $\overline{\overline{H}}_{\mathrm{ir}}'$, Eq. (2.28b) is of dimension four or higher, since it vanishes for $h = 0$, and the linear contribution to H' is correctly given by (2.28a) with $\overline{T}'$ satisfying Eq. (2.18). The terms in (2.28b) are clearly irrelevant, but in contrast to (2.17c), there are an infinite number of terms at each order in h. A different but equally valid division of terms in (2.30) would be obtained, for example, if $\overline{T}'$ and $\bar{h}'$ were calculated from the renormalization of H_1, Eq. (2.22b), rather than the renormalization of H_0, but once again the difference would only involve irrelevant operators.

3. ANISOTROPIC MODELS

In this section we discuss quadratic symmetry breaking represented by the Hamiltonian[3]

$$H = \int d^dx \left\{ \left(-\frac{1}{2T} \right) \left[(\partial_\mu \pi)^2 + (\partial_\mu \sigma)^2 + g\pi_1^2 \right] \right\} \tag{3.1}$$

where π_1 is the first component[3] of the vector π. In the present case we know from physical arguments that for $g > 0$ the system crosses over to a fixed point characteristic of the $O(n-1)$ symmetry, whereas for $g < 0$ the system becomes Ising-like. It is not sufficient to discuss the renormalized Hamiltonian for small g' (as we did for h in Section 2.2), if we wish to describe the full crossover. We shall therefore introduce a nonliner renormalization which is the same as that of Nelson and Pelcovits,[(4)] but we shall discuss the systematics of the renormalization in order to verify the consistency of the procedure. Let us represent the vector field π as $(\pi_1, \tilde{\pi})$, where $\tilde{\pi}$ has $(n-2)$ components. Then (3.1) may be written as[3]

$$H[\tilde{\pi}, \pi_1] = \int d^dx \left\{ \left(-\frac{1}{2T} \right) \left[(\partial_\mu \tilde{\pi})^2 + (\partial_\mu \pi_1)^2 + g\pi_1^2 + \left(1 - \tilde{\pi}^2 - \pi_1^2\right)^{-1} \left(\tilde{\pi} \cdot \partial_\mu \tilde{\pi} + \pi_1 \partial_\mu \pi_1\right)^2 \right] - \tfrac{1}{2}\rho \ln\left(1 - \tilde{\pi}^2 - \pi_1^2\right) \right\} \tag{3.2}$$

in terms of which the thermal average of an operator $X[\tilde{\pi}, \pi_1]$ is given by

$$\langle X \rangle = Z^{-1} \int D\tilde{\pi} \int D\pi_1 X[\tilde{\pi}, \pi_1] \exp\{ H[\tilde{\pi}, \pi_1] \} \tag{3.3}$$

where Z is the partition function and $\mathfrak{D}\tilde{\pi}$ denotes functional integration. For $g = 0$ the renormalization of (3.2) must be identical to that of (2.3), i.e., one can verify that the coefficients of $(\partial_\mu\tilde{\pi})^2$, $(\partial_\mu\pi_1)^2$, $(\tilde{\pi}\cdot\partial_\mu\tilde{\pi})^2$, $2(\tilde{\pi}\cdot\partial_\mu\tilde{\pi})(\pi_1\partial_\mu\pi_1)$, etc., are the same at each stage, up to terms containing four or more derivatives. For $g > 0$, we represent H' in the following form:

$$H'[\tilde{\pi},\pi_1] = \overline{H}' + \overline{H}'_{\text{ir}} \tag{3.4}$$

$$\overline{H}' = \int d^dx\left\{\left[-\frac{1}{2T'(g)}\right]\left[(\partial_\mu\tilde{\pi})^2 + (\partial_\mu\pi_1)^2 + g'\pi_1^2 + \left(1-\tilde{\pi}^2-\pi_1^2\right)^{-1}\left(\tilde{\pi}\cdot\partial_\mu\tilde{\pi}+\pi_1\partial_\mu\pi_1\right)^2\right] - \tfrac{1}{2}\rho\ln\left(1-\tilde{\pi}^2-\pi_1^2\right)\right\} \tag{3.5a}$$

$$\overline{H}'_{\text{ir}} = \int d^dx\left\{\sum_{m=2}^{\infty}A_m(g)\pi_1^{2m} + \sum_{m=1}^{\infty}\sum_{k=0}^{\infty}\left[B_{mk}(g)(\tilde{\pi}\cdot\partial_\mu\tilde{\pi})^2 + E_{mk}(g)\tilde{\pi}^2(\partial_\mu\tilde{\pi})^2\right]\pi_1^{2m}\tilde{\pi}^{2k} + \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\left[C_{mk}(g)(\tilde{\pi}\cdot\partial_\mu\tilde{\pi})(\pi_1\partial_\mu\pi_1) + D_{mk}(g)(\pi_1\partial_\mu\pi_1)^2\right]\times\pi_1^{2m}\tilde{\pi}^{2k} + O\left(\partial_\mu^4\right)\right\} \tag{3.5b}$$

The functions T', ζ_{π_1}, g', and $\zeta_{\tilde{\pi}}$ are determined[4] by studying the renormalization of $(\partial_\mu\tilde{\pi})^2$, $(\partial_\mu\pi_1)^2$, π_1^2, and $(\tilde{\pi}\cdot\partial_\mu\tilde{\pi})^2$, respectively, and lead to nonlinear recursion relations of the form

$$T'(g) = b^{-\epsilon}T\left[1 + \frac{T}{2\pi}\,\frac{\left[(n-3)(1+g/\Lambda^2)+1\right]}{1+g/\Lambda^2}\ln b\right] \tag{3.6}$$

$$g'(g) = b^2 g\left[1 + \frac{T}{2\pi}\,\frac{1}{1+g/\Lambda^2}\ln b\right] \tag{3.7}$$

The renormalized coefficients of terms such as $(\pi_1\partial_\mu\pi_1)^2$, $\pi_1^2(\tilde{\pi}\cdot\partial_\mu\tilde{\pi})^2$, etc., are not equal to $T'(g)$, but the difference is made up by the coefficients in the added term (3.5b). The coefficients of $(\partial_\mu\tilde{\pi})^2$, $(\tilde{\pi}\cdot\partial_\mu\tilde{\pi})^2$, and $\tilde{\pi}^{2m}(\tilde{\pi}\cdot\partial_\mu\tilde{\pi})^2$ $(m = 1, \ldots, \infty)$ are identical, on the other hand, and equal to $T'(g)$.

The important point to note about $\overline{H}'_{\text{ir}}$, Eq. (3.5b), is that it vanishes in the limit $g\to 0$, and remains bounded for large g. Indeed, the coefficients

$A_m(g)$ can be calculated[9] and are equal to

$$A_m(g) = m^{-1}[g/(\Lambda^2+g)]^m \tag{3.8}$$

while inspection of the perturbation theory[9] shows that the coefficients B, C, and D in (3.5b) are bounded. Moreover, it is also clear that $\overline{H}'_{\text{ir}}$ vanishes for $\pi_1 = 0$.

As discussed by Nelson and Pelcovits,[4] the recursion relations (3.6) and (3.7) display a crossover between the $O(n)$ fixed point

$$g^* = 0, \qquad T^* = 2\pi\epsilon/(n-2) \tag{3.9}$$

and the $O(n-1)$ fixed point

$$g^* = \infty, \qquad T^* = 2\pi\epsilon/(n-3) \tag{3.10}$$

Let us verify that the full Hamiltonian (3.4)–(3.5) yields the correct physics in both limits. For $g' \to 0$ the discussion of Section 2.1 applies and the terms in (3.5b) containing two or fewer derivatives must vanish identically. The crossover exponent

$$\lambda_g = 2 - 2\epsilon/(n-2) \tag{3.11}$$

found by Nelson and Pelcovits[4] depends on the terms of the form $g\pi_1^2$ in (3.5a), and has no contribution from (3.5b), since the latter involves terms with higher powers of g [see Eq. (3.8)].

When the transformation (3.7) is iterated repeatedly, the coefficient g' grows and eventually reaches the fixed point (3.10). In that limit the procedure of Ref. 4 may be justified by the following argument: for large g the quantity $g\pi_1^2$ in (3.5a) grows without bounds while all the other terms in (3.5a) and (3.5b) remain bounded, for fixed π_1 and $\tilde{\pi}$. [The boundedness of (3.5b) is an important element in the argument.] Therefore, any thermal average such as (3.3) can be simplified,[12] i.e.,

$$\langle X \rangle = Z^{-1} \int D\tilde{\pi}\, X[\tilde{\pi}, \pi_1 = 0] \exp\{H[\tilde{\pi}, \pi_1 = 0]\} \tag{3.12}$$

since the configurations in (3.4) with $\pi_1 \neq 0$ have very large energy. It follows that the effective Hamiltonian is

$$H[\tilde{\pi}] = \int d^dx \left\{ \left[-\frac{1}{2T(g=\infty)} \right] \left[(\partial_\mu \tilde{\pi})^2 + (1-\tilde{\pi}^2)^{-1} (\tilde{\pi} \cdot \partial_\mu \tilde{\pi})^2 \right] \right.$$
$$\left. - \tfrac{1}{2}\rho \ln(1-\tilde{\pi}^2) + O(\partial_\mu^4) \right\} \tag{3.13}$$

i.e., it is precisely the Hamiltonian for an $O(n-1)$ system.

For intermediate values of g, we can represent H' in terms of the parameters $g'(g)$ and $T'(g)$ given by (3.6)–(3.7), though the precise breakup in (3.4)–(3.5) is rather arbitrary. This is because although $\overline{H}'_{\text{ir}}$

contains only irrelevant operators, $\overline{H}'$ contains both relevant and irrelevant terms. By including irrelevant operators in $\overline{H}'$ the latter is made to have the same form as the starting Hamiltonian H. This procedure allows us to define renormalized values of T and g unambiguously for intermediate values of g.

Let us discuss briefly the case $g < 0$, when a crossover to Ising-like behavior is expected. The representation (3.2) is still an acceptable form for the renormalization group for small $|g|$. Now, however, as $|g|$ grows, configurations in (3.4) with $\pi_1 \neq 0$ become more and more favored so the contribution of the correction (3.5) cannot be neglected. Since we expect[11] the critical temperature to be of order $T_c \sim [-\ln|g|]$, the small-T perturbation theory is no longer valid, and an accurate description of the crossover is beyond the scope of the present methods.

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APPENDIX: EVALUATION OF DIAGRAMS WITH DERIVATIVE COUPLING

Pelcovits[13] has discussed the evaluation of Feynmann graphs for the 4-point interaction $(\pi \cdot \partial_\mu \pi)^2$ with derivative coupling. It turns out, however, that his approach gives incorrect results in the presence of h or g. In order to illustrate this point, we consider the second and fourth diagrams in Fig. 1b. Since these diagrams involve fewer than two derivatives on the external legs, the contributions of order ∂_μ^2 have to be obtained in a rather nontrivial way. If $\mathbf{p}$ and $\mathbf{p}+\mathbf{q}$ denote the momenta on the two internal propagators, we have

$$\begin{aligned} \Lambda/b &< |\mathbf{p}| < \Lambda \\ \Lambda/b &< |\mathbf{p}+\mathbf{q}| < \Lambda \end{aligned} \tag{A.1}$$

Hence, the $\mathbf{p}$ integration appearing in the computation of the diagrams has to be done in an elliptical shell. For $h = 0$ and $g = 0$, this elliptical q-dependent boundary of $\mathbf{p}$ does not contribute to the q^2 terms, as can be seen from the following arguments: From dimensional analysis, it is obvious that the integrals appearing in the evaluation of the diagrams are

$$q^2 \int_{\text{el}} d^2p/p^2 \tag{A.2a}$$

$$\int_{\text{el}} (\mathbf{p} \cdot \mathbf{q})\, d^2p/p^2 \tag{A.2b}$$

$$\int_{\text{el}} d^2p \tag{A.2c}$$

where "el" denotes the elliptical integration domain which depends on q via Eq. (A.1). However, to obtain terms proportional to q^2, it is clear that the limits of integration in (A.2a) can be considered independent of q. To show that the integrals (A.2b) and (A.2c) also do not contribute to the q^2 term, let us define

$$\mathbf{p}' = \mathbf{p} + \mathbf{r} \tag{A.2d}$$

where $\mathbf{p}'$ lies in a spherical shell $\Lambda/b \leqslant |\mathbf{p}'| \leqslant \Lambda$. The integral (A.2b) becomes

$$\int_{\text{sph}} \frac{(\mathbf{p}' - \mathbf{r}) \cdot \mathbf{q}}{(\mathbf{p}' - \mathbf{r})^2}\, d^2p' = 0 \tag{A.2b$'$}$$

since it can be viewed as representing an electric field inside a charged hollow cylinder. The change of variable in (A.2d) implies that (A.2c) will also be independent of q.

For $h \neq 0$, $g \neq 0$, the evaluation of the diagrams also involves the following integrals [in addition to those given by (A.2)]:

$$hq^2 \int d^2p/p^4 \tag{A.3a}$$

$$h \int (\mathbf{q} \cdot \mathbf{p})\, d^2p/p^4 \tag{A.3b}$$

$$h \int d^2p/p^2 \tag{A.3c}$$

The analogy with electrostatics no longer exists in this case, and hence the contribution to q^2-dependent terms from (A.3b) and (A.3c) is nonzero. However, it is not easy to calculate these integrals under the constraint (A.1). Instead, if we redistribute the momenta on the propagators to $\mathbf{p} - \mathbf{q}/2$ and $\mathbf{p} + \mathbf{q}/2$, (A.1) reduces to

$$\begin{aligned} &\Lambda/b < |\mathbf{p} - \mathbf{q}/2| < \Lambda \\ &\Lambda/b < |\mathbf{p} + \mathbf{q}/2| < \Lambda \end{aligned} \tag{A.4}$$

The above relation is invariant under $\mathbf{p} \to -\mathbf{p}$. Hence integrals of the kind (A.3b) are always zero. Also, the q^2 contribution from (A.3c) can be easily obtained using the constraint (A.4).[9]

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